$\mathbb{R}^3$: $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ orthonormal Dul Sum

6.3 – Gram-Schmidt Process

orthogonal

$\mathbb{R}^2$: $\{(3,4,0), (4,-3,0), (0,0,1)\}$

orthonormal: $\{(\frac{3}{5}, \frac{4}{5}, 0), (\frac{4}{5}, -\frac{3}{5}, 0), (0,0,1)\}$

Definition: A set of two or more vectors in a real inner product space is said to be **orthogonal** if all pairs of distinct vectors in the set are orthogonal. An orthogonal set in which each vector has norm 1 is said to be **orthonormal**.

Just as with the Euclidean inner product, we can normalize a vector to obtain a unit vector: $\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v}$

Theorem 6.3.1 If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal set of nonzero vectors in an inner product space, then S is linearly independent.

pf: Suppose $k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_n\vec{v}_n = \vec{0}$ and consider $\langle k_1\vec{v}_1 + k_2\vec{v}_2 + \cdots + k_n\vec{v}_n, \vec{v}_i\rangle$, $1 \le i \le n$.

$= \langle \vec{0}, \vec{v}_i\rangle = 0.$

But $\underbrace{\langle k_1\vec{v}_1, \vec{v}_i\rangle}_{=0} + \underbrace{\langle k_2\vec{v}_2, \vec{v}_i\rangle}_{=0} + \cdots + \langle k_i\vec{v}_i, \vec{v}_i\rangle + \cdots + \underbrace{\langle k_n\vec{v}_n, \vec{v}_i\rangle}_{=0}$

$= \langle k_i\vec{v}_i, \vec{v}_i\rangle = k_i\langle \vec{v}_i, \vec{v}_i\rangle$

$\Rightarrow k_i\langle \vec{v}_i, \vec{v}_i\rangle = 0$ since $\vec{v}_i \ne 0$,

$k_i = 0$, $1 \le i \le n$.

S is linearly independent.

Definition: In an inner product space, a basis comprising orthogonal vectors is an **orthogonal basis**, and a basis comprising orthonormal vectors is an **orthonormal basis**.

#6 Show that the column vectors of A form an orthogonal basis for the column space of A with respect to the Euclidean inner product, and then find an orthonormal basis for that column space. — dot product

$$A = \begin{array}{c} \begin{array}{ccc} \vec{c}_1 & \vec{c}_2 & \vec{c}_3 \end{array} \\ \begin{bmatrix} 1/5 & -1/2 & 1/3 \\ 1/5 & 1/2 & 1/3 \\ 1/5 & 0 & -2/3 \end{bmatrix} \end{array}$$

$\vec{c}_1 \cdot \vec{c}_2 = (1/5, 1/5, 1/5) \cdot (-1/2, 1/2, 0) = -\frac{1}{10} + \frac{1}{10} = 0$

$\vec{c}_1 \cdot \vec{c}_3 = (1/5, 1/5, 1/5) \cdot (1/3, 1/3, -2/3) = \frac{1}{15} + \frac{1}{15} - \frac{2}{15} = 0$

$\vec{c}_2 \cdot \vec{c}_3 = -\frac{1}{6} + \frac{1}{6} = 0$

this is an orthogonal basis.

$\cancel{\|\vec{c}_1\| = \sqrt{\frac{1}{25} + \frac{1}{25} + \frac{1}{25}}}$ $\vec{c}_1 \to$ use $\|(1,1,1)\| = \sqrt{3}$

$\vec{c}_2$: $(-1/2, 1/2, 0) \to \|(-1, 1, 0)\| = \sqrt{2}$

$\vec{c}_3$: use $(1, 1, -2) \to \|(1, 1, -2)\| = \sqrt{6}$

$\left\{ \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right) \right\}$

is an orthonormal basis.

Theorem 6.3.2

a) If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal basis for an inner product space V, and if $\mathbf{u}$ is any vector in V, then

$$\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \ldots \frac{\langle \mathbf{u}, \mathbf{v}_n \rangle}{\|\mathbf{v}_n\|^2}\mathbf{v}_n$$

b) If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthonormal basis for an inner product space V, and if $\mathbf{u}$ is any vector in V, then $\mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \ldots + \langle \mathbf{u}, \mathbf{v}_n \rangle \mathbf{v}_n$.

Pf: a) Let $\vec{u} \in V$. Then since S is a basis for V, $\exists\, c_i \ni \vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$.

$\langle \vec{u}, \vec{v}_i \rangle$, $1 \le i \le n$ is $\langle c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n, \vec{v}_i \rangle$

$= c_i \langle \vec{v}_i, \vec{v}_i \rangle$ since S is orthogonal.

$\Rightarrow \langle \vec{u}, \vec{v}_i \rangle = c_i \langle \vec{v}_i, \vec{v}_i \rangle \Rightarrow c_i = \dfrac{\langle \vec{u}, \vec{v}_i \rangle}{\|\vec{v}_i\|^2}$

b) If $\|\vec{v}_i\|^2 = 1$, then $c_i = \langle \vec{u}, \vec{v}_i \rangle$.

#8 Use Theorem 6.3.2(b) to express the vector $\mathbf{u} = (3, -7, 4)$ as a linear combination of the vectors $\mathbf{v}_1 = \left(-\frac{3}{5}, \frac{4}{5}, 0\right)$, $\mathbf{v}_2 = \left(\frac{4}{5}, \frac{3}{5}, 0\right)$, $\mathbf{v}_3 = (0, 0, 1)$.

$\vec{u} = (\vec{u} \cdot \vec{v}_1)\vec{v}_1 + (\vec{u} \cdot \vec{v}_2)\vec{v}_2 + (\vec{u} \cdot \vec{v}_3)\vec{v}_3$

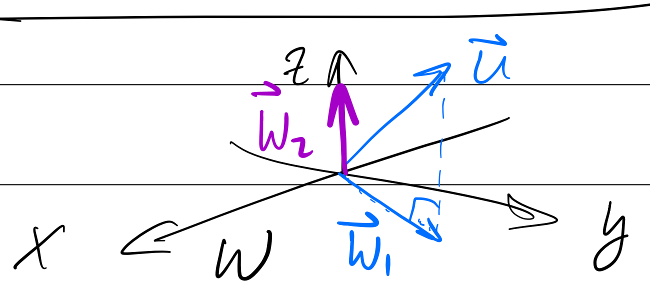
$\vec{u} = \left(-\frac{9}{5} - \frac{28}{5}\right)\vec{v}_1 + \left(\frac{12}{5} - \frac{21}{5}\right)\vec{v}_2 + 4\vec{v}_3$

$$\boxed{\vec{u} = -\frac{37}{5}\vec{v}_1 - \frac{9}{5}\vec{v}_2 + 4\vec{v}_3}$$

#12 Find the coordinate vector $(\mathbf{u})_S$ for the vector $\mathbf{u}$ and the basis S that were given in Exercise 8.

$(\vec{u})_S = \left(-\frac{37}{5}, -\frac{9}{5}, 4\right)$

$W \subset \mathbb{R}^3$



Theorem 6.3.3 Projection Theorem (generalization of Theorem 3.3.2)
If W is a finite-dimensional subspace of an inner product space V, then every vector $\mathbf{u}$ in V can be expressed in exactly one way as $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$ where $\mathbf{w}_1$ is in W and $\mathbf{w}_2$ is in $W^{\perp}$.

Definition: If W is a finite-dimensional subspace of an inner product space V and a vector $\mathbf{u}$ in V is expressed as $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ is in W and $\mathbf{w}_2$ is in $W^{\perp}$, then $\mathbf{w}_1$ is called the **orthogonal projection of u on W** and is denoted by $\mathbf{w}_1 = \text{proj}_W \mathbf{u}$ and $\mathbf{w}_2$ is called the **orthogonal projection of u on W**$^{\perp}$ and is denoted by $\mathbf{w}_2 = \text{proj}_{W^{\perp}} \mathbf{u}$. The vector $\mathbf{w}_2$ is also called the **component of u orthogonal to W**.

Theorem 6.3.4 Let W be a finite-dimensional subspace of an inner product space V.

a) If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is an orthogonal basis for W and $\mathbf{u}$ is any vector in V, then

$$\text{proj}_W \mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\langle \mathbf{u}, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \ldots \frac{\langle \mathbf{u}, \mathbf{v}_r \rangle}{\|\mathbf{v}_r\|^2}\mathbf{v}_r$$

b) If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is an orthonormal basis for W and $\mathbf{u}$ is any vector in V, then

$$\text{proj}_W \mathbf{u} = \langle \mathbf{u}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{u}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \ldots + \langle \mathbf{u}, \mathbf{v}_r \rangle \mathbf{v}_r$$

pf: From Thm 6.3.3, we have $\vec{u} = \vec{w}_1 + \vec{w}_2$.

$$\text{proj}_W \vec{u} = \vec{w}_1 = \frac{\langle \vec{w}_1, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 + \frac{\langle \vec{w}_1, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 + \cdots \frac{\langle \vec{w}_1, \vec{v}_r \rangle}{\|\vec{v}_r\|^2}\vec{v}_r$$

Since $\vec{w}_2 \in W^{\perp}$, $\langle \vec{w}_2, \vec{v}_i \rangle = 0$, $1 \le i \le r$

$$\text{proj}_W \vec{u} = \vec{w}_1 = \frac{\langle \vec{w}_1 + \vec{w}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 + \cdots + \frac{\langle \vec{w}_1 + \vec{w}_2, \vec{v}_r \rangle}{\|\vec{v}_r\|^2}\vec{v}_r$$

$$= \frac{\langle \vec{u}, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 + \cdots + \frac{\langle \vec{u}, \vec{v}_r \rangle}{\|\vec{v}_r\|^2}\vec{v}_r$$

If $\|\vec{v}_i\|^2 = 1$, $1 \le i \le r$, then $\vec{w}_1 = \langle \vec{u}, \vec{v}_1 \rangle \vec{v}_1 + \cdots + \langle \vec{u}, \vec{v}_r \rangle \vec{v}_r$

#24 The vectors $\mathbf{v}_1 = (0,\ 1,\ -4,\ -1)$ and $\mathbf{v}_2 = (3,\ 5,\ 1,\ 1)$ are orthogonal with respect to the Euclidean inner product on R^4. Find the orthogonal projection $\mathbf{w}_1$ of $\mathbf{b} = (1,\ 2,\ 0,\ -2)$ on the subspace W spanned by these vectors. [Extension of the exercise] Then find $\mathbf{w}_2$ in $W^{\perp}$ such that $\mathbf{b} = \mathbf{w}_1 + \mathbf{w}_2$.

$$\vec{w}_1 = \text{proj}_W \vec{b} = \frac{\vec{b}\cdot\vec{v}_1}{\|\vec{v}_1\|^2}\vec{v}_1 + \frac{\vec{b}\cdot\vec{v}_2}{\|\vec{v}_2\|^2}\vec{v}_2$$

$$= \frac{4}{18}(0,1,-4,-1) + \frac{11}{36}(3,5,1,1)$$

$$\text{proj}_W \vec{b} = \left(\frac{11}{12}, \frac{7}{4}, -\frac{7}{12}, \frac{1}{12}\right)$$

$$\vec{b} = \vec{w}_1 + \vec{w}_2 \Rightarrow \vec{w}_2 = \vec{b} - \vec{w}_1$$

$\vec{w}_2 \in W^{\perp}$

$$\vec{w}_2 = (1,2,0,-2) - \left(\frac{11}{12}, \frac{7}{4}, -\frac{7}{12}, \frac{1}{12}\right) = \left(\frac{1}{12}, \frac{1}{4}, \frac{7}{12}, -\frac{25}{12}\right)$$

#25 The vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are orthonormal with respect to the Euclidean inner product on R^4. Find the orthogonal projection of $\mathbf{b} = (1,\ 2,\ 0,\ -1)$ onto the subspace W spanned by these vectors.

$\mathbf{v}_1 = \left(0, \frac{1}{\sqrt{18}}, -\frac{4}{\sqrt{18}}, -\frac{1}{\sqrt{18}}\right)$, $\mathbf{v}_2 = \left(\frac{3}{6}, \frac{5}{6}, \frac{1}{6}, \frac{1}{6}\right)$, $\mathbf{v}_3 = \left(\frac{1}{\sqrt{18}}, 0, \frac{1}{\sqrt{18}}, -\frac{4}{\sqrt{18}}\right)$

$$\text{proj}_W \vec{b} = (\vec{b}\cdot\vec{v}_1)\vec{v}_1 + (\vec{b}\cdot\vec{v}_2)\vec{v}_2 + (\vec{b}\cdot\vec{v}_3)\vec{v}_3$$

$$= \frac{3}{18}(0,1,-4,-1) + \frac{12}{36}(3,5,1,1) + \frac{5}{18}(1,0,1,-4)$$

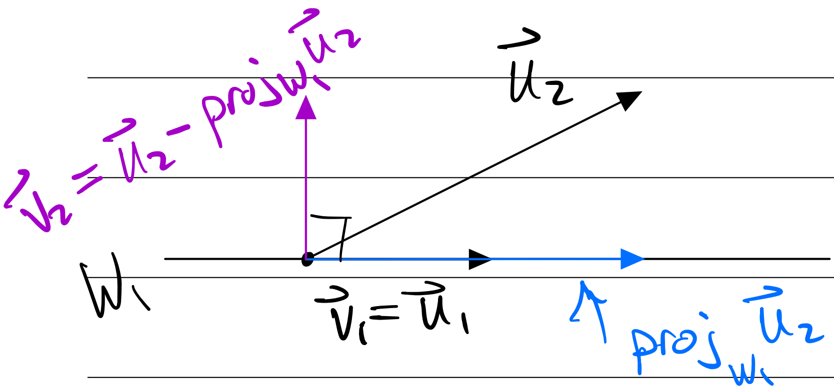
$$= \left(\frac{23}{18}, \frac{11}{6}, -\frac{1}{18}, -\frac{17}{18}\right)$$

Theorem 6.3.5 (proof outlines the Gram-Schmidt process)
Every nonzero finite-dimensional inner product space has an orthonormal basis.

This is a constructive proof.

<u>pf</u>: Let W be any nonzero finite-dimensional subspace of an inner product space and suppose $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_r\}$ is any basis for W. We will produce an orthogonal basis $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_r\}$ for W that can be normalized to form an orthonormal basis.

Let $\vec{v}_1 = \vec{u}_1$ and W_1 be the space spanned by $\vec{v}_1$.

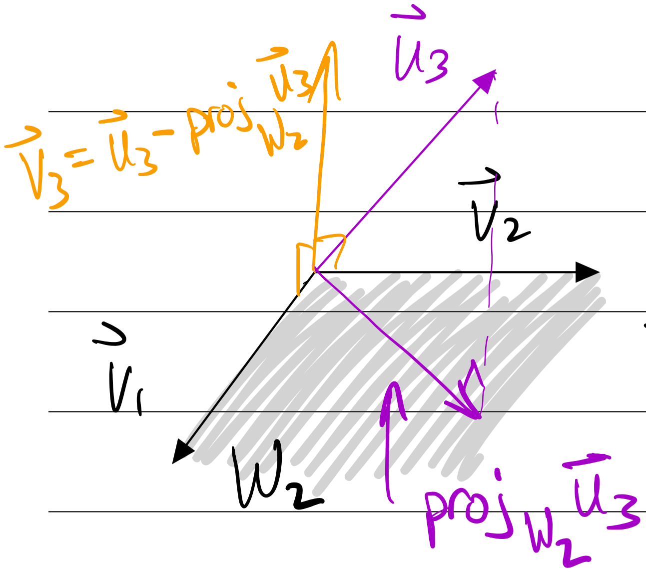


Let $\vec{v}_2 = \vec{u}_2 - \text{proj}_{W_1}\vec{u}_2$

$$\vec{v}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1$$

We know $\vec{v}_2 \neq \vec{0}$, because if $\vec{v}_2 = \vec{0}$, then $\vec{u}_2$ is a scalar multiple of $\vec{v}_1 = \vec{u}_1$, but $\{\vec{u}_i\}$ is a basis. By construction, $\vec{v}_2 \perp \vec{v}_1$.

Now let W_2 be the space spanned by $\vec{v}_1$ & $\vec{v}_2$.



$$\vec{v}_3 = \vec{u}_3 - \text{proj}_{W_2}\vec{u}_3$$

$$\vec{v}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_3, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2$$

$\vec{v}_3$ is orthogonal to W_2, so it is orthogonal to $\vec{v}_1$, $\vec{v}_2$.

Likewise, we can find a vector $\vec{v}_4$ that is orthogonal to $\vec{v}_1, \vec{v}_2, \vec{v}_3$:

$$\vec{v}_4 = \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} - \frac{\langle \vec{u}_4, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2}\vec{v}_2 - \frac{\langle \vec{u}_4, \vec{v}_3 \rangle}{\|\vec{v}_3\|^2}\vec{v}_3$$

and so on until we have r vectors.

Normalizing each $\vec{v}_i$ results in an orthonormal basis.

#31 Let R^4 have the Euclidean inner product. Use the Gram-Schmidt process to transform the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ into an orthonormal basis.
$\mathbf{u}_1 = (0, 2, 1, 0)$, $\mathbf{u}_2 = (1, -1, 0, 0)$, $\mathbf{u}_3 = (1, 2, 0, -1)$, $\mathbf{u}_4 = (1, 0, 0, 1)$

$$\vec{v}_1 = \vec{u}_1 = (0,2,1,0)$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\vec{u}_2 \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\vec{v}_1 = (1,-1,0,0) - \frac{-2}{5}(0,2,1,0)$$

$$\vec{v}_2 = \left(1, -\frac{1}{5}, \frac{2}{5}, 0\right)$$

Use $\vec{v}_2{}' = (5,-1,2,0)$

* check for orthogonality as you go* $\vec{v}_1 \cdot \vec{v}_2{}' = 0$

$$\vec{v}_3 = \vec{u}_3 - \frac{\vec{u}_3 \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\vec{u}_3 \cdot \vec{v}_2{}'}{\|\vec{v}_2{}'\|^2}\vec{v}_2{}'$$

$$\vec{v}_3 = (1,2,0,-1) - \frac{4}{5}(0,2,1,0) - \frac{3}{30}(5,-1,2,0)$$

$$\vec{v}_3 = \left(\frac{1}{2}, \frac{1}{2}, -1, -1\right). \text{ Use } \vec{v}_3{}' = (1,1,-2,-2)$$

$\vec{v}_3{}' \cdot \vec{v}_1 = 0$ $\quad$ $\vec{v}_3{}' \cdot \vec{v}_2{}' = 0$

$$\vec{v}_4 = \vec{u}_4 - \frac{\langle \vec{u}_4, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{u}_4, \vec{v}_2{}' \rangle}{\|\vec{v}_2{}'\|^2}\vec{v}_2{}' - \frac{\langle \vec{u}_4, \vec{v}_3{}' \rangle}{\|\vec{v}_3{}'\|^2}\vec{v}_3{}'$$

$$= (1,0,0,1) - \frac{0}{5}\vec{v}_1 - \frac{5}{30}(5,-1,2,0) - \frac{-1}{10}(1,1,-2,-2)$$

$$\vec{v}_4 = \left(\frac{4}{15}, \frac{4}{15}, -\frac{8}{15}, \frac{4}{5}\right). \text{ Use } \cancel{\vec{v}_4{}' = (4,4,-8,12)}$$

Use $\vec{v}_4{}' = (1,1,-2,3)$

orthogonal basis: $\{\overset{\vec{v}_1}{(0,2,1,0)}, \overset{\vec{v}_2'}{(5,-1,2,0)}, \overset{\vec{v}_3'}{(1,1,-2,-2)}, \overset{\vec{v}_4'}{(1,1,-2,3)}\}$

orthonormal basis:

$\{(0, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0), (\frac{5}{\sqrt{30}}, -\frac{1}{\sqrt{30}}, \frac{2}{\sqrt{30}}, 0), (\frac{1}{\sqrt{10}}, \frac{1}{\sqrt{10}}, -\frac{2}{\sqrt{10}}, -\frac{2}{\sqrt{10}}), (\frac{1}{\sqrt{15}}, \frac{1}{\sqrt{15}}, -\frac{2}{\sqrt{15}}, \frac{3}{\sqrt{15}})\}$

Theorem 6.3.6 (inner product space analog of Theorem 4.6.5 (b))
If W is a finite-dimensional inner product space, then:
a) Every orthogonal set of nonzero vectors in W can be enlarged to an orthogonal basis for W.
b) Every orthonormal set of nonzero vectors in W can be enlarged to an orthonormal basis for W.

Example 6.3.9 Legendre Polynomials
Let the vector space P_2 have the inner product $\langle \mathbf{p}, \mathbf{q} \rangle = \int_{-1}^{1} p(x)q(x)dx$. Apply the Gram-Schmidt process to transform the standard basis $\{1, x, x^2\}$ for P_2 into an orthogonal basis $\{\phi_1(x), \phi_2(x), \phi_3(x)\}$.

$\varphi_1 = 1$

$\varphi_2 = x - \frac{\langle x, 1 \rangle}{\|1\|^2} 1$ but $\langle 1, x \rangle = \int_{-1}^{1} x\,dx = 0$

$\varphi_2 = x$

$\varphi_3 = x^2 - \frac{\langle x^2, 1 \rangle}{\|1\|^2} 1 - \frac{\langle x^2, x \rangle}{\|x\|^2} x$

$\langle x^2, 1 \rangle = \int_{-1}^{1} x^2 dx = \frac{2}{3}$, $\|1\|^2 = \int_{-1}^{1} 1\,dx = 2$

$\varphi_3 = x^2 - \frac{1}{3}$ $\langle x^2, x\rangle = \int_{-1}^{1} x^3 dx = 0$

$\{\varphi_1, \varphi_2, \varphi_3\} = \{1, x, x^2 - \frac{1}{3}\}$

Consider φ_3: $a(x^2 - \frac{1}{3})$

We want a so that $a\varphi_3(1) = 1$

$a(1 - \frac{1}{3}) = 1 \Rightarrow \frac{2}{3}a = 1 \Rightarrow a = \frac{3}{2}$

$\frac{3}{2}\varphi_3 = \frac{3}{2}(x^2 - \frac{1}{3}) = \frac{1}{2}(3x^2 - 1)$

$\rightarrow \{1, x, \frac{1}{2}(3x^2 - 1)\}$

Definition: The first three **Legendre polynomials** are 1, x, and $\frac{1}{2}\left(3x^2 - 1\right)$. They result from transforming the standard basis for P_2 with the inner product $\langle \mathbf{p}, \mathbf{q}\rangle = \int_{-1}^{1} p(x)q(x)dx$ into an orthogonal basis for P_2 via the Gram-Schmidt process, scaled so they have a value of 1 at $x = 1$.